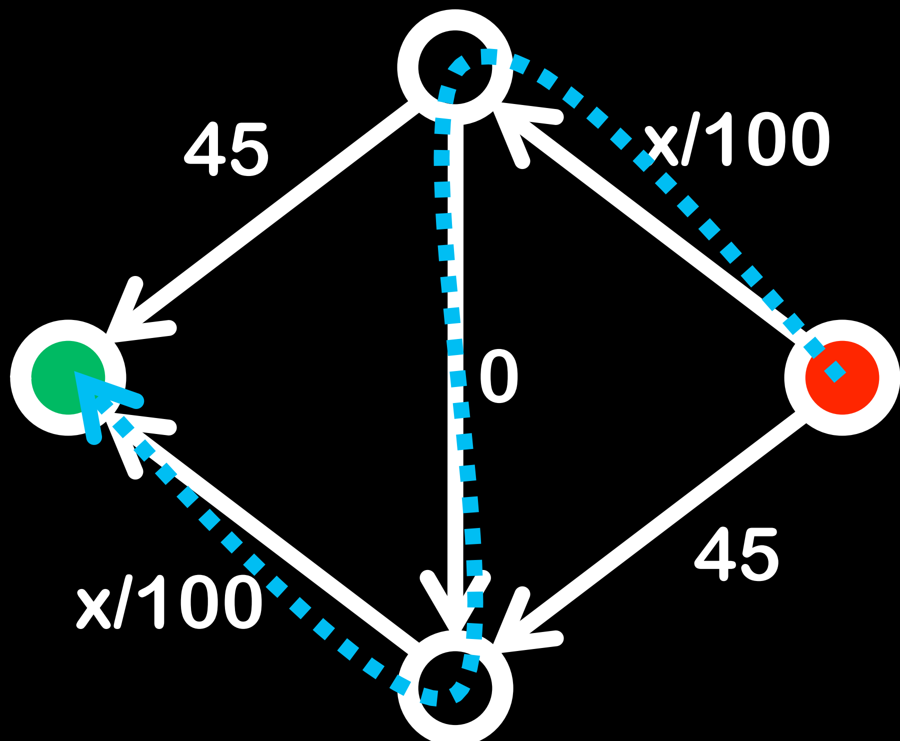
15-396

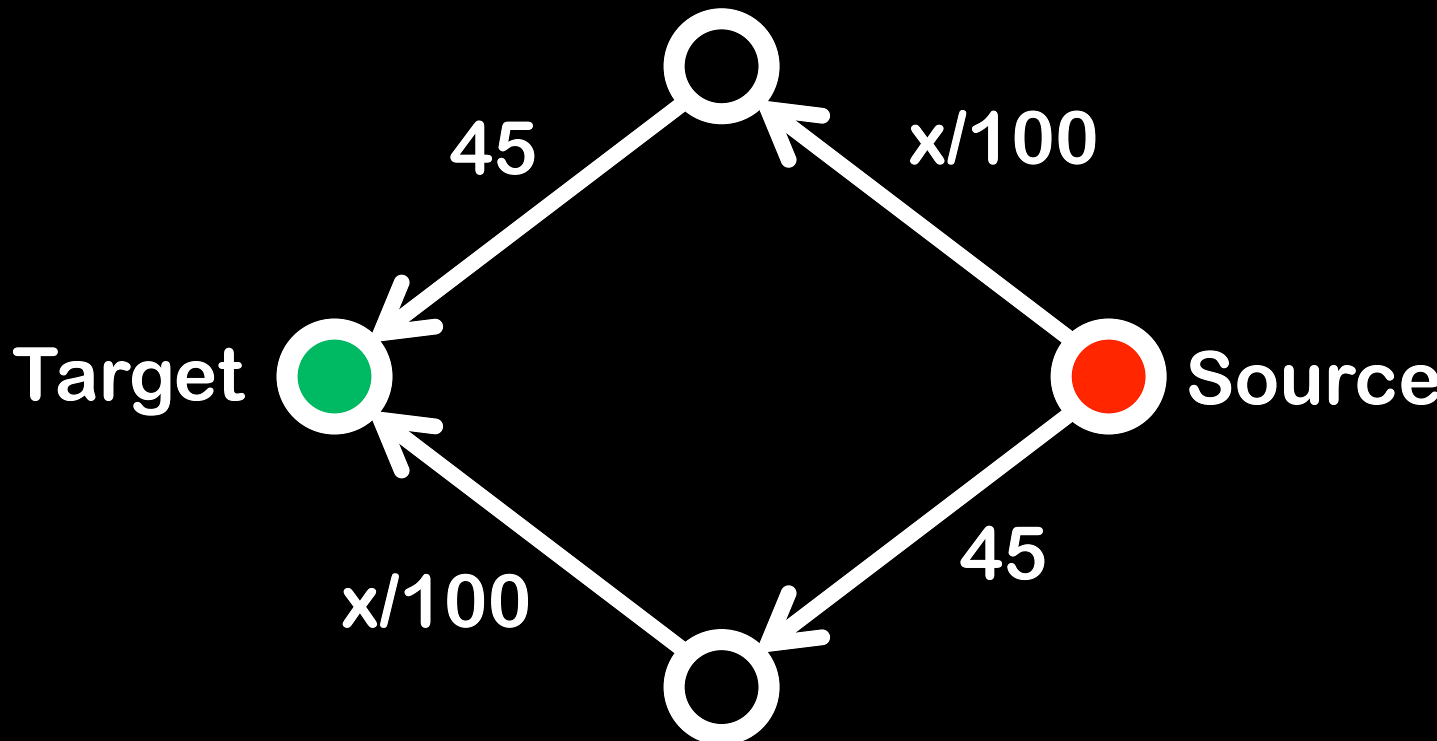
Science of teh Interwebs

Modeling Network Traffic Using Game Theory

Lecture 10 (October 11, 2011)



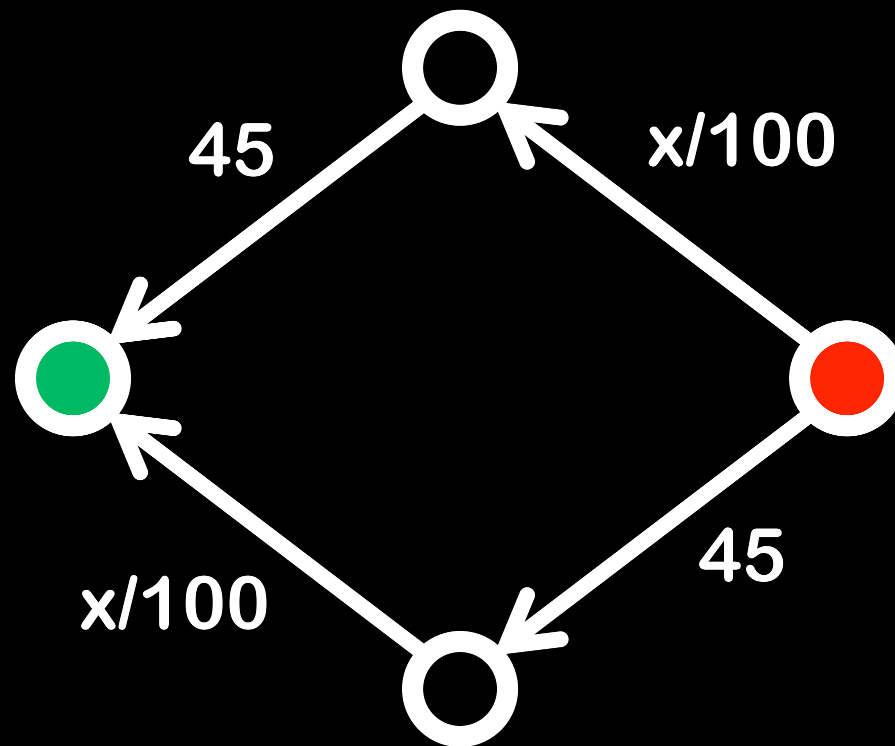
Transportation Network



If 4,000 cars take the upper route, the travel time of each is 85

If 4,000 cars split evenly between the upper and lower routes, the travel time is 65

This is a game in which the cars are the players, and the strategies are the possible routes

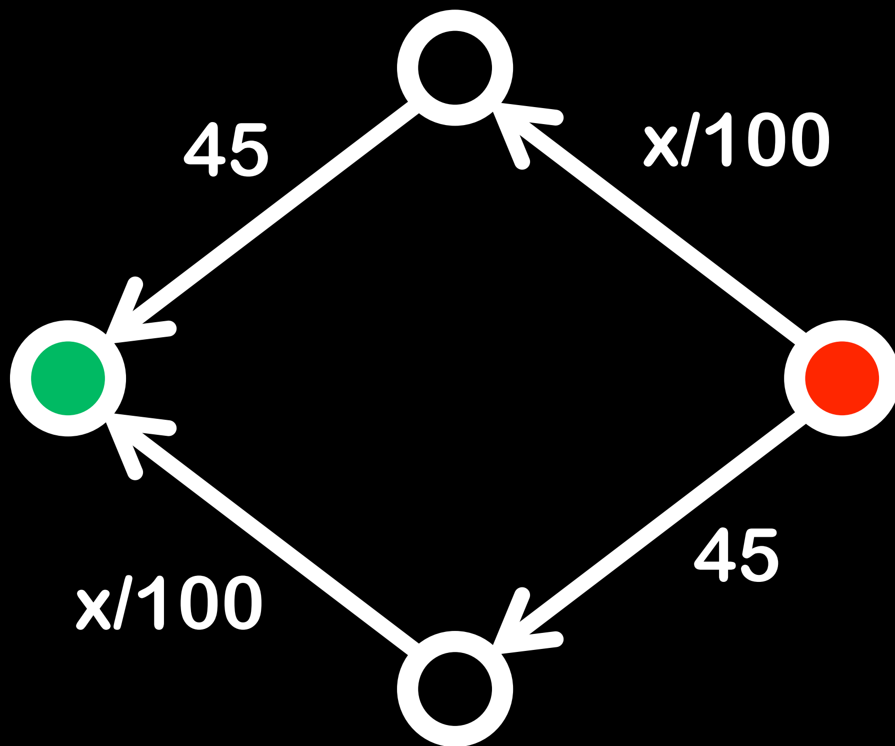


Is there a dominant strategy?

No, because either route has the potential to be the best choice for a player if all cars take the other route

What is a Nash Equilibrium?

Any list of strategies in which the cars balance themselves equally between the two routes

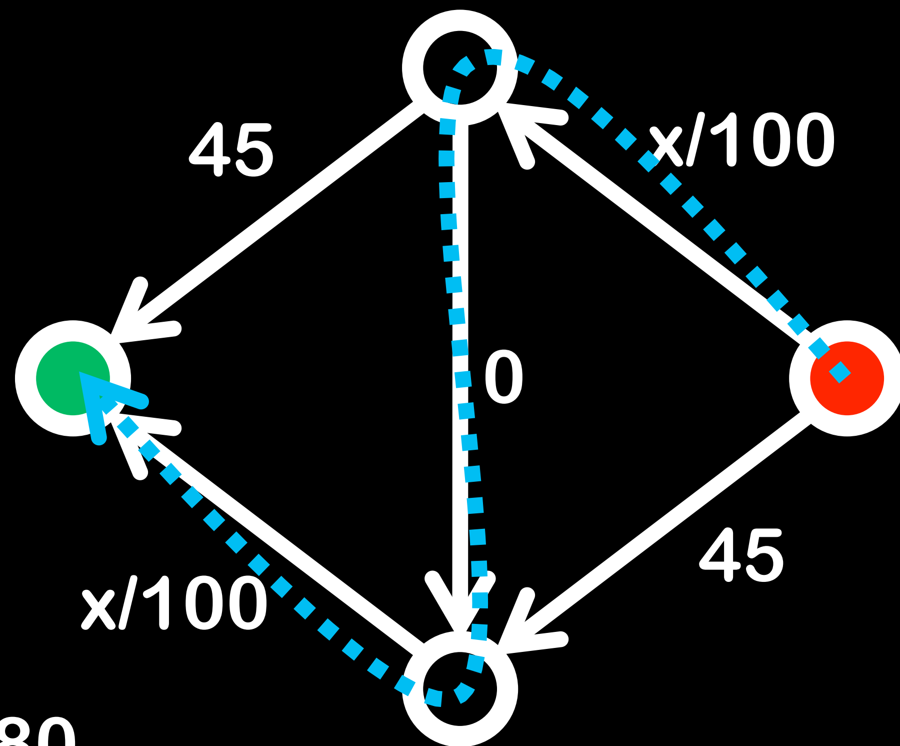


With an even balance between the routes, no car has an incentive to switch over to the other route

These are the only Nash equilibria: if the traffic is uneven, cars have an incentive to switch

This ought to be faster, but it's not!

The Nash Equilibrium is for everybody to take the middle edge



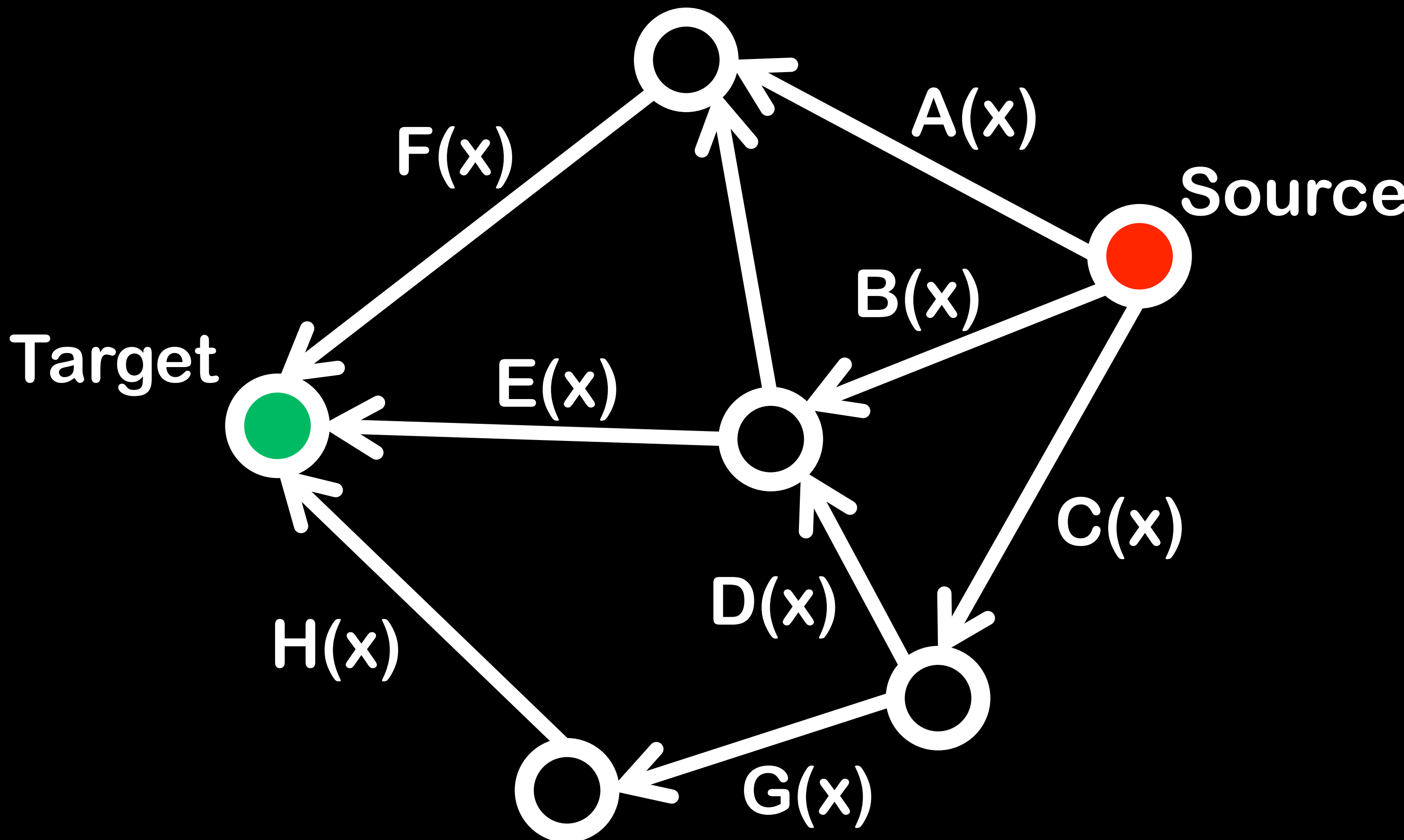
And the travel time is 80

This is a Nash Equilibrium because no driver can benefit from changing their route

In fact, taking the middle edge is the only dominant strategy for everybody

Braess's Paradox: Adding resources to a transportation network can sometimes hurt performance at equilibrium

This has been observed in real life!



If the latency functions are linear, adding edges can never make the total travel time at equilibrium worse by more than a factor of 4/3

Braess's Paradox is an example that network traffic at equilibrium is not socially optimal

How far from optimal is traffic at equilibrium?

Definitions

The network can be any directed graph

Different drivers may have different starting points and destinations

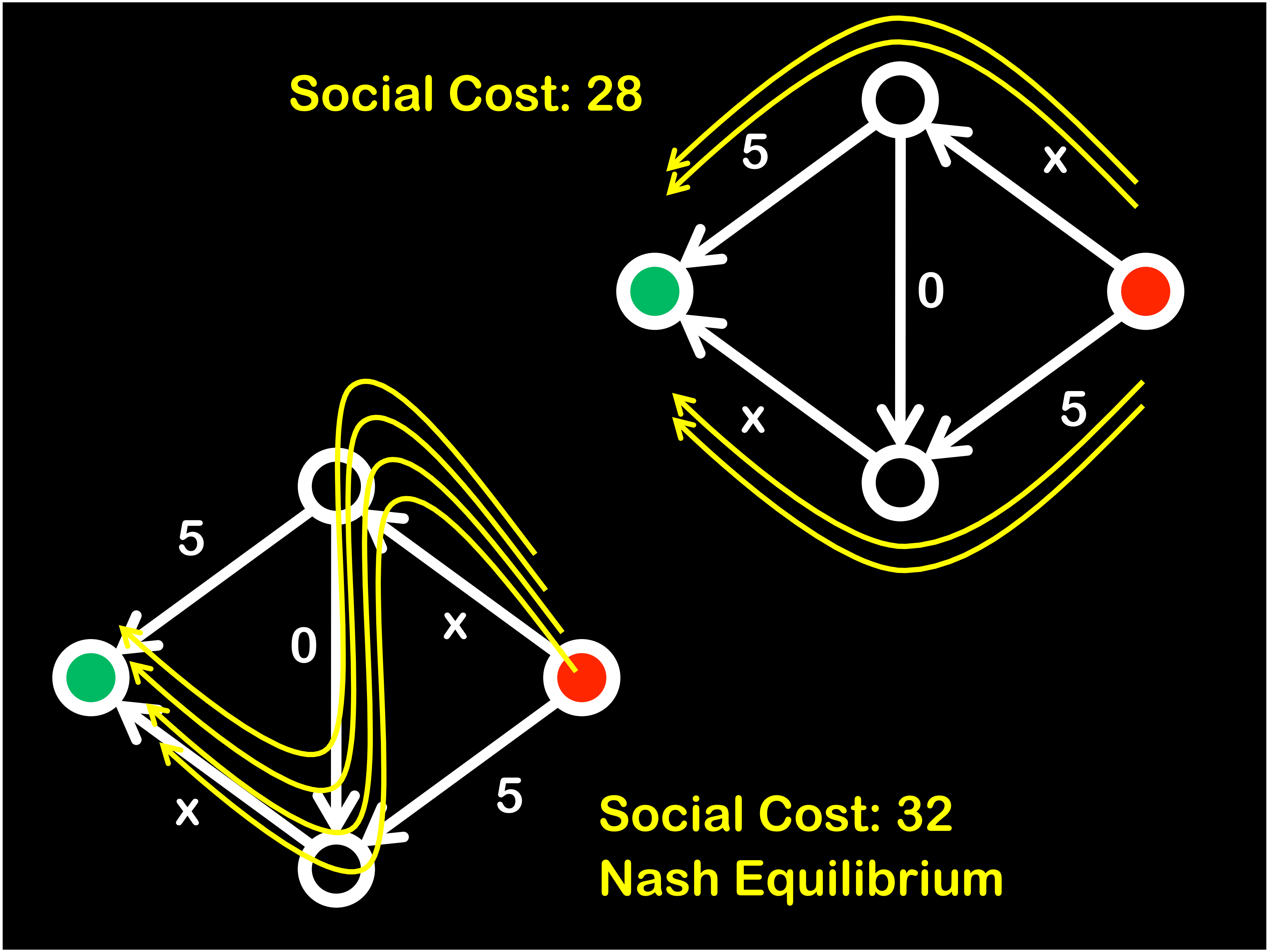
S_j = starting point for car j

T_j = target for car j

Strategies for car j are possible paths from S_j to T_j

Each edge e has a travel function $L_e(x) = a_e x + b_e$ ($a_e, b_e \geq 0$)

The social cost of a given traffic pattern is the sum of travel times incurred by each driver



Social Cost: 28
5
x
0
x
5
5
0
x
x
5
Social Cost: 32
Nash Equilibrium

Finding Equilibria

Start from any traffic pattern

If it's an equilibrium, stop

Else, there is at least one driver whose best response, given what everyone else is doing is some alternate path

Pick one such driver and switch them to this alternate path

Repeat

This procedure is called "Best Response Dynamics"

Does it Terminate?

We will define a progress measure

If an edge has x drivers on it:

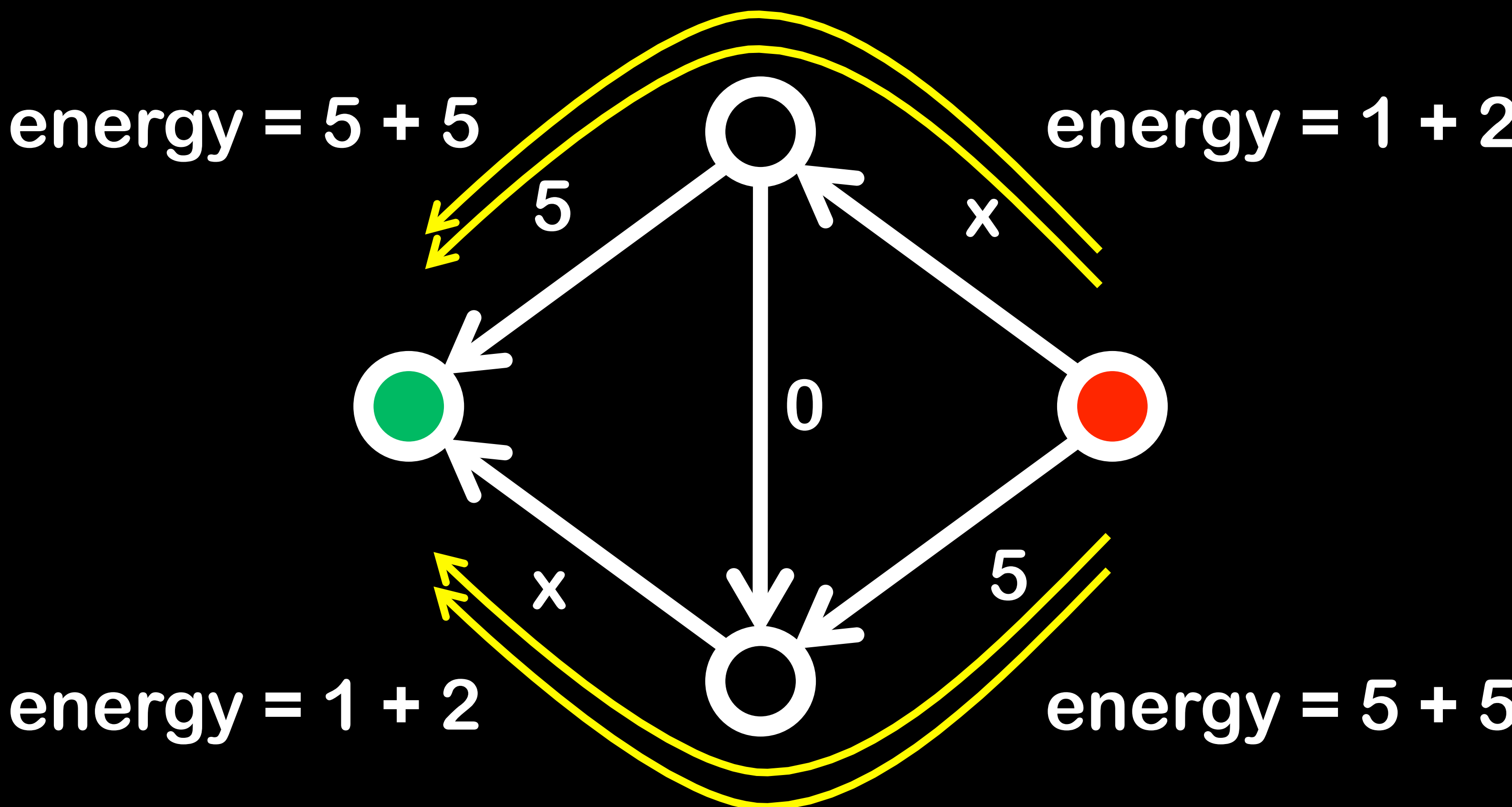
$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$

Else:

$$\text{Energy}(e) = 0$$

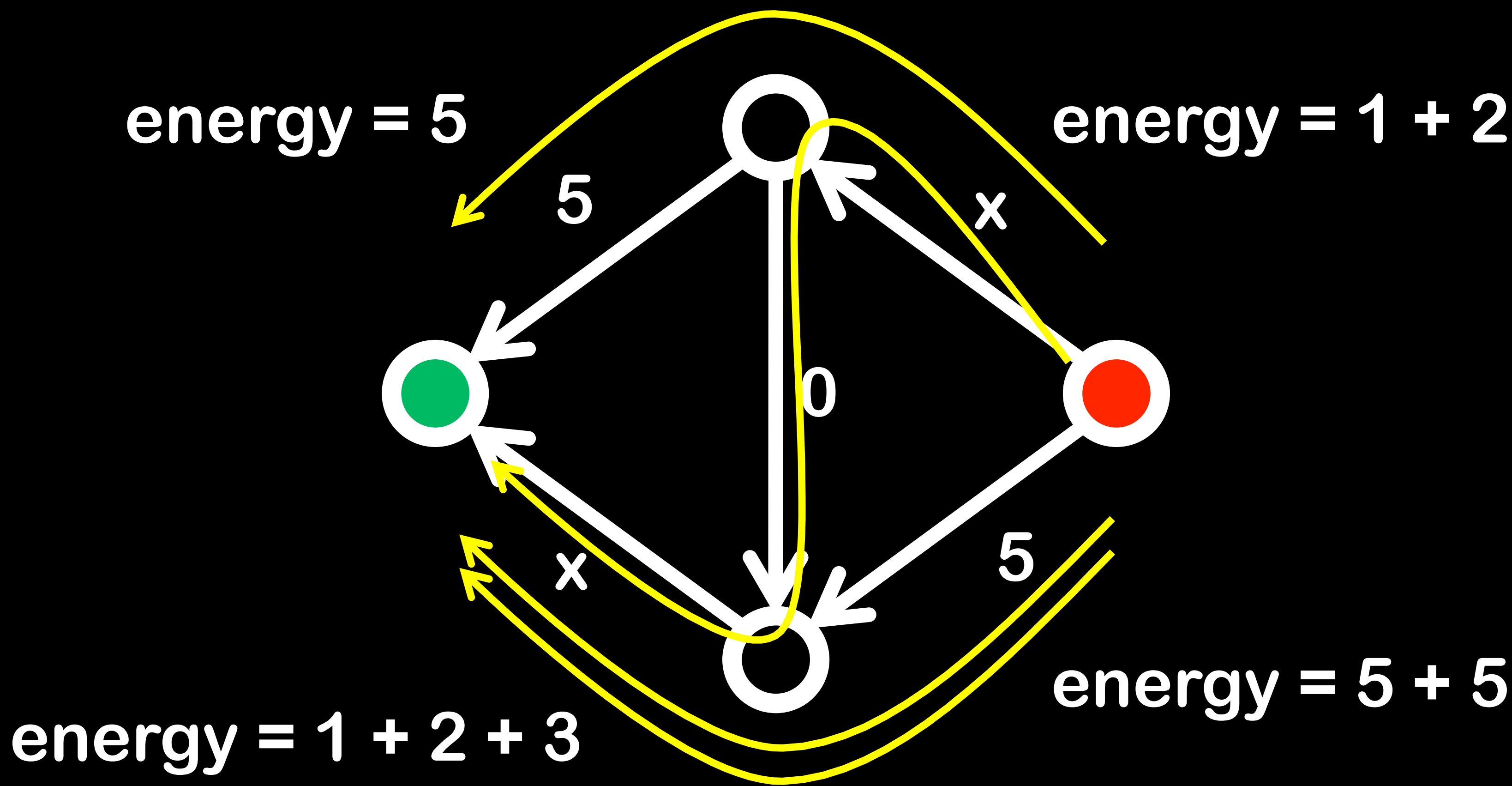
The potential energy of a traffic pattern is the sum of potential energies of all edges

$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$



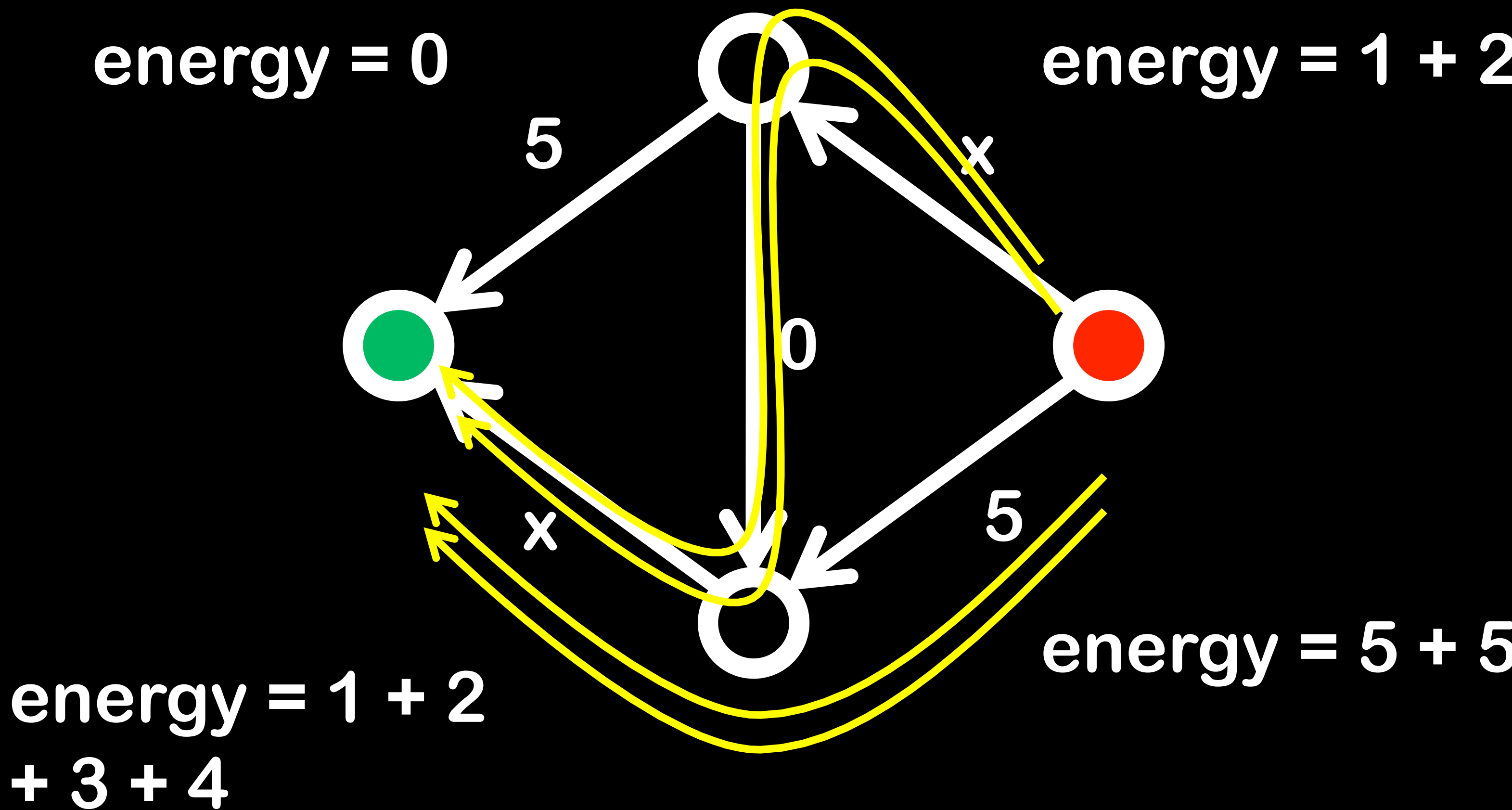
Sum = 26

$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$



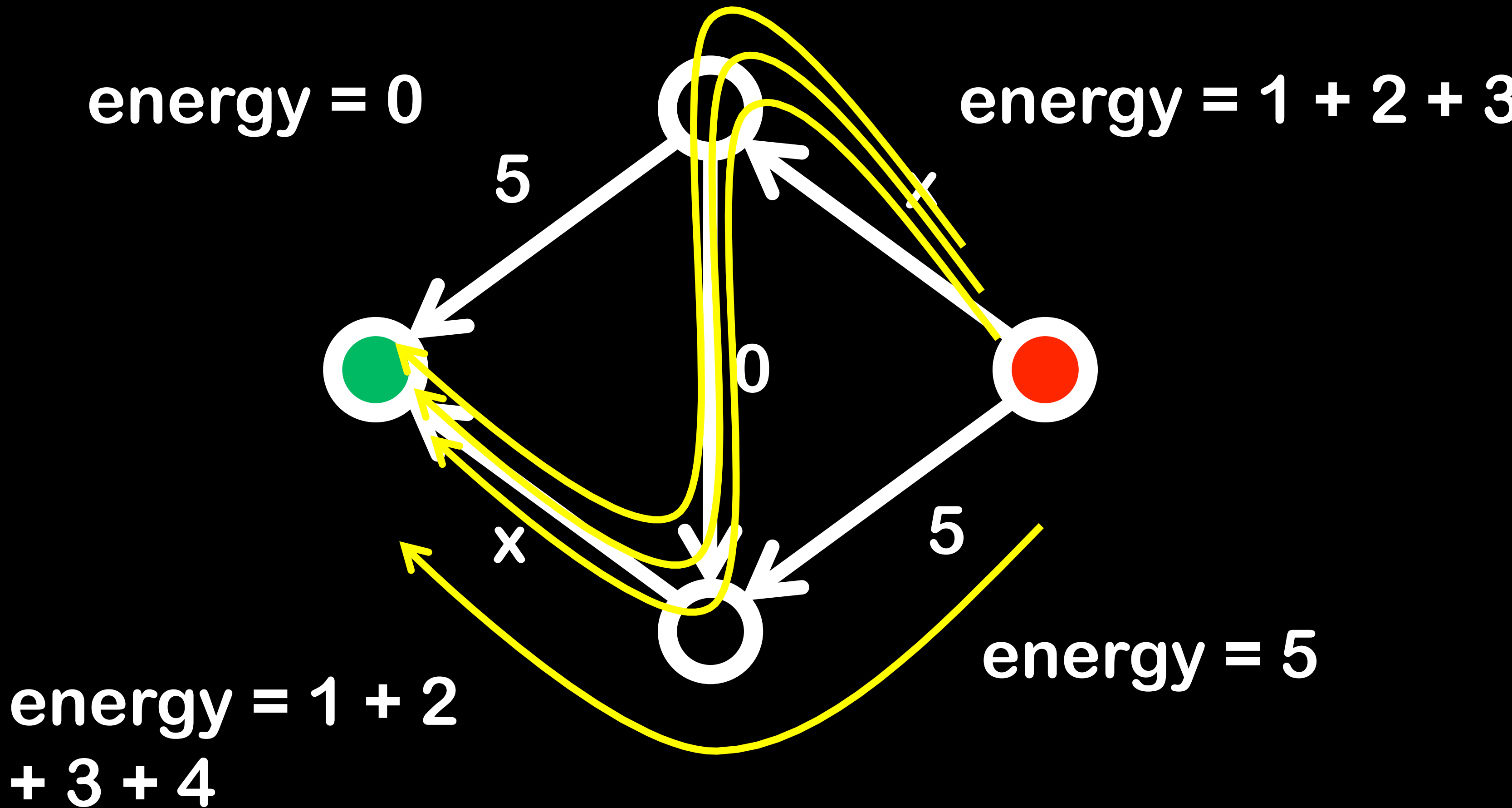
Sum = 24

$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$



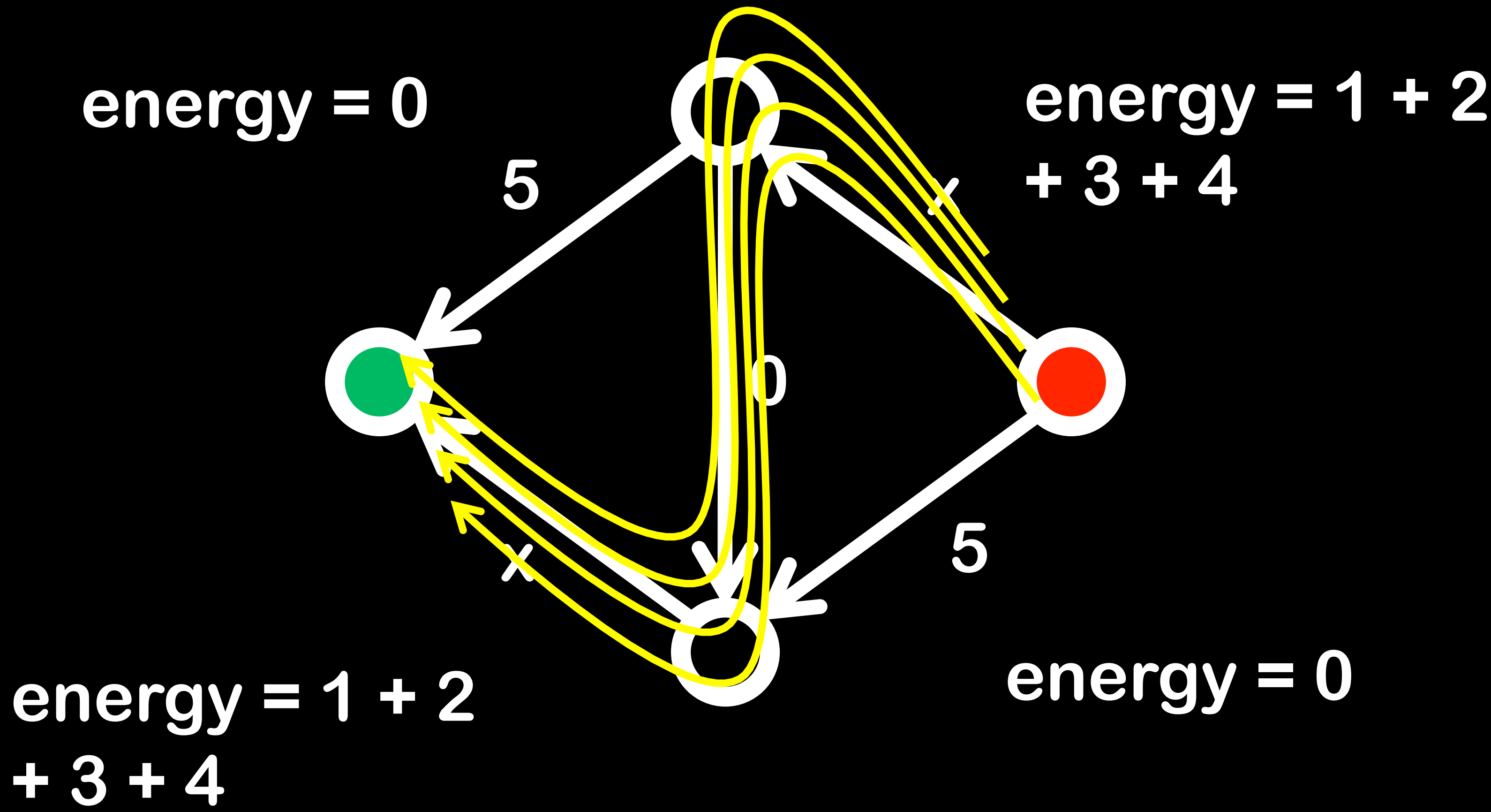
Sum = 23

$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$



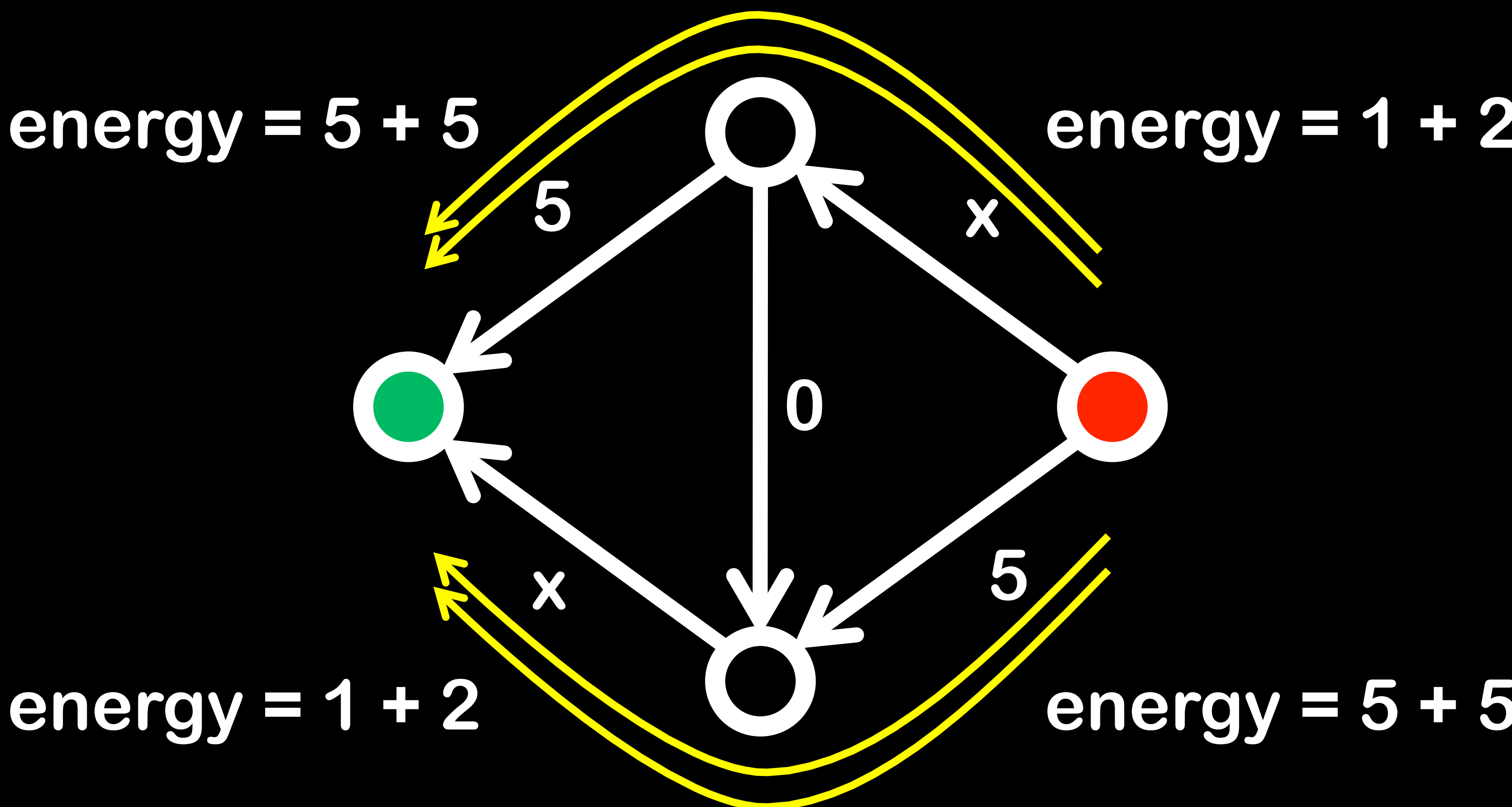
Sum = 21

$$Energy(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$



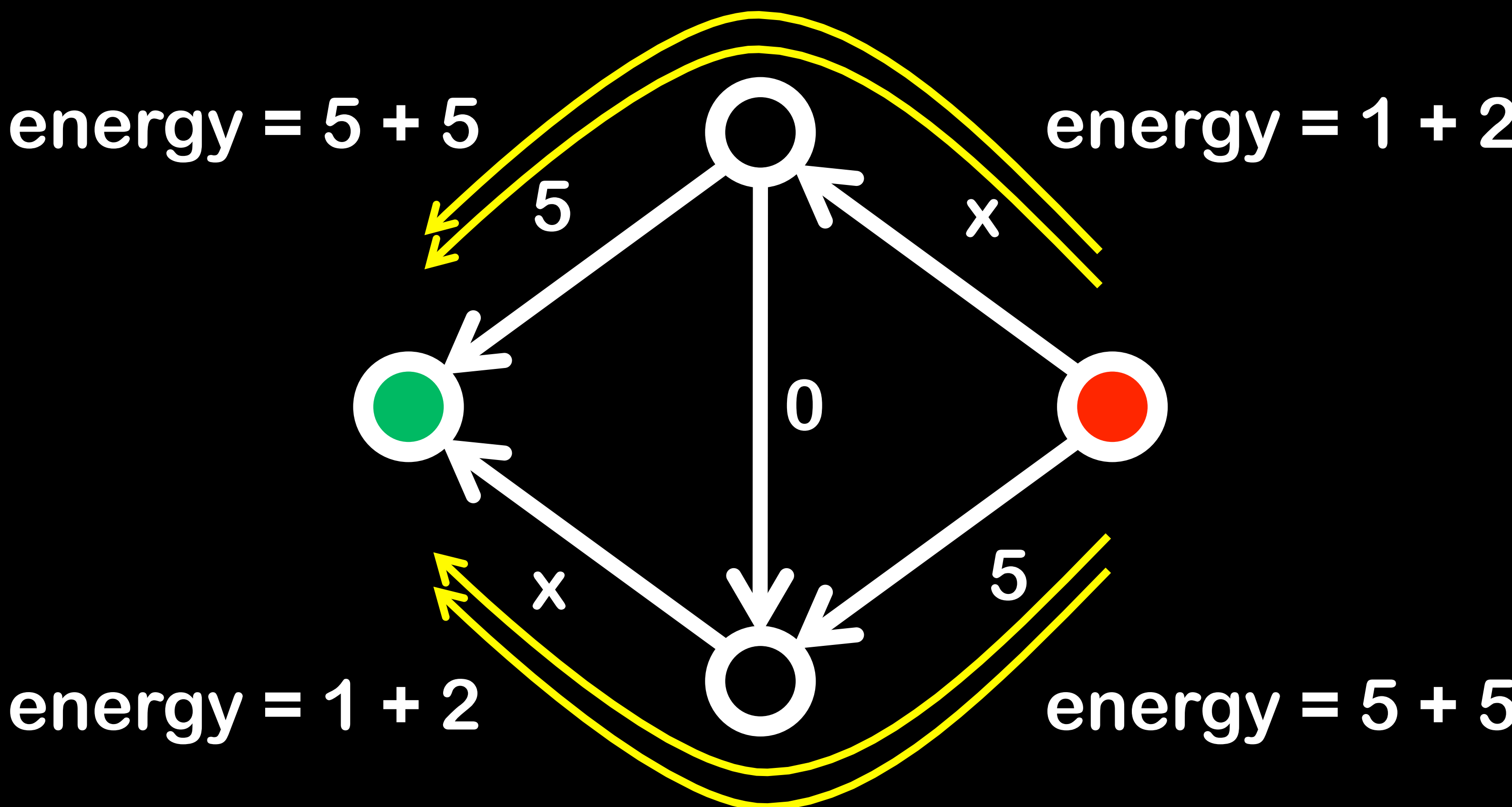
Sum = 20

Theorem: Each step of best response dynamics causes the potential energy of the current traffic pattern to strictly decrease



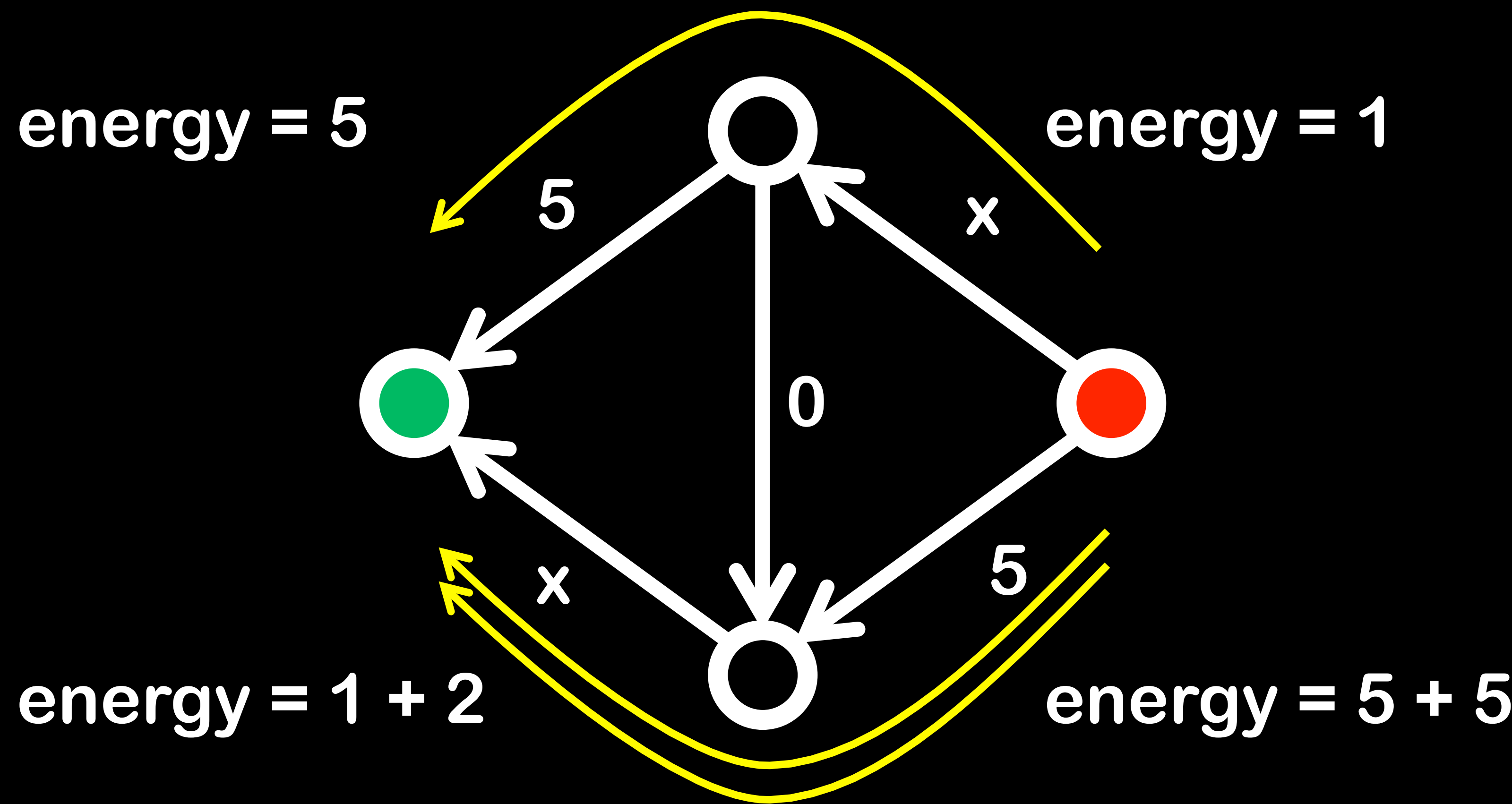
Can think of changing a route as first removing it and then adding a new one

Theorem: Each step of best response dynamics causes the potential energy of the current traffic pattern to strictly decrease



E = 26

Theorem: Each step of best response dynamics causes the potential energy of the current traffic pattern to strictly decrease

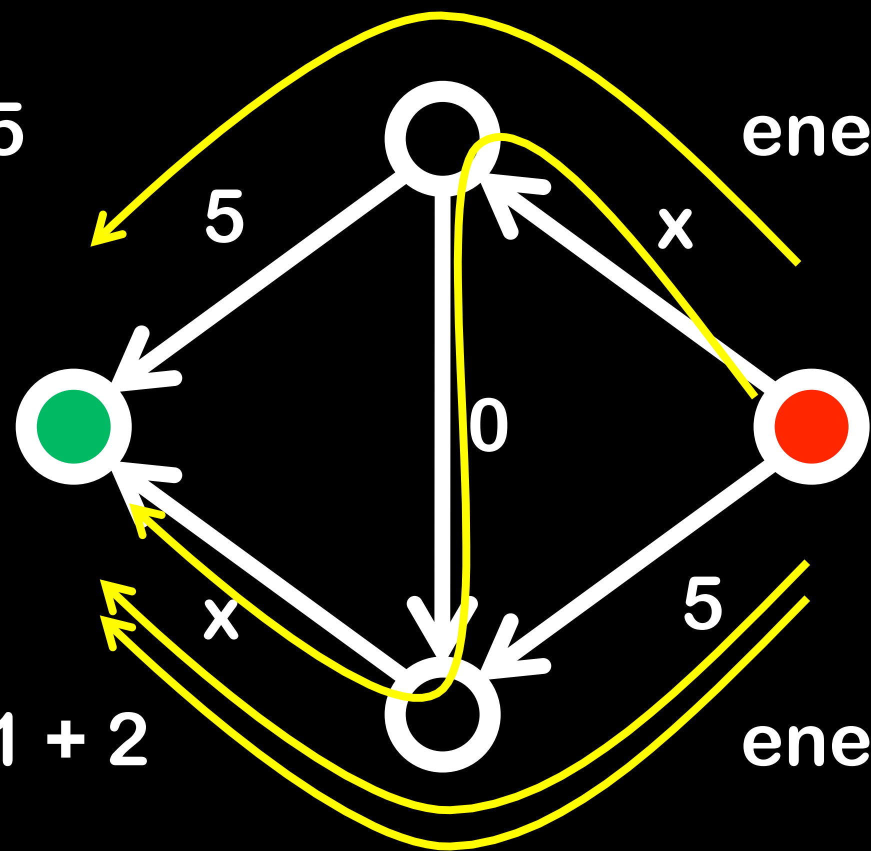


$$E = 26 \rightarrow 19$$

Theorem: Each step of best response dynamics causes the potential energy of the current traffic pattern to strictly decrease

energy = 5

energy = 1 + 2



energy = 1 + 2 + 3

energy = 5 + 5

E = 26 → 19 → 24

Theorem: Each step of best response dynamics causes the potential energy of the current traffic pattern to strictly decrease

Energy on edge e with x drivers:

$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$

Energy on edge e with x-1 drivers:

$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x-1)$$

The energy released when a driver abandons his current path is exactly equal to the travel time the driver was experiencing

The energy added to the system when a driver adopts a new path is exactly equal to the travel time the driver now experiences

Theorem: Each step of best response dynamics causes the potential energy of the current traffic pattern to strictly decrease

Corollary: Best response dynamics must eventually halt

How Bad is Traffic at Equilibrium?

Energy on edge e with x drivers:

$$\text{Energy}(e) = L_e(1) + L_e(2) + \ldots + L_e(x)$$

Total time spent by all drivers on that edge:

$$\text{Total-Travel-Time}(e) = \underbrace{L_e(x) + L_e(x) + \ldots + L_e(x)}_{x \text{ terms}}$$

$\text{Energy}(e) \leq \text{Total-Travel-Time}(e)$

Recall that $L_e(x) = a_e x + b_e$

$$L_e(1) + \ldots + L_e(x) = a_e(1+2+\ldots x) + b_e x$$

$$= a_e x(x+1)/2 + b_e x$$

$$= x(a_e(x+1)/2 + b_e)$$

$$\geq (1/2)x(a_e x + b_e)$$

$$= (1/2)\text{Total-Travel-Time}(e)$$

(1/2)Total-Travel-Time(e) ≤ Energy(e) ≤ Total-Travel-Time(e)

If Z is a traffic pattern:

$$(1/2)\text{Social-Cost}(Z) \leq \text{Energy}(Z) \leq \text{Social-Cost}(Z)$$

If we start from a socially optimal traffic pattern Z and end in an equilibrium pattern Z':

$$\begin{aligned}\text{Social-Cost}(Z') &\leq 2\ \text{Energy}(Z') \\ &\leq 2\ \text{Energy}(Z) \\ &\leq 2\ \text{Social-Cost}(Z)\end{aligned}$$

How Bad is Traffic at Equilibrium?

At worst, twice as bad as optimal traffic

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