Assignment 4 Solution

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1 Little Ditties

1.

TRUE By the definition of dominant strategy, any strategy of A is a best response to B. Player B will strive to play the best response to A always. A pair of strategy where each is a best response to each other is called a Nash equilibrium. Therefore, if (s_A, s_B) are both best responses to each other, then a pure strategy Nash equilibrium must exist.

2.

Pure strategy equilibrium is (D,L) and (U,R).

(D,L) is an equilibrium because when player A plays D, player B is better off playing L; likewise, if player B plays L, player A is better off playing D. (U,R) is an equilibrium because when player A plays U, player B is better off playing R; likewise, if player B plays R, player A is better off playing U.

Randomized Nash equilibrium is $(\frac{1}{2}, \frac{1}{2})$.

Let p be the probability that player A plays U and q be the probability that player B plays L. Then, we have the following four equations:

$$\begin{aligned}
\text{E(player A playing U)} &= 1q + 4(1-q) &= 4 - 3q \\
\text{E(player A playing D)} &= 3q + 2(1-q) &= 2 + q \\
\text{E(player B playing L)} &= 1p + 3(1-p) &= 3 - 2p \\
\text{E(player B playing R)} &= 2p + 2(1-p) &= 2
\end{aligned}$$

Player A is willing to randomize only if the expected payoff of U and D are equal, i.e.

$$\begin{aligned} 4-3q &= 2+q \\ 2 &= 4q \\ q &= \frac{1}{2} \end{aligned}$$

Likewise, B is willing to randomize only if the expected payoff of L and R are equal, i.e.

$$\begin{aligned} 3-2p &= 2 \\ 1 &= 2p \\ p &= \frac{1}{2} \end{aligned}$$

2 Seller's Revenue

(a)

When there are two bidders, there are four possible pairs of values (0,0), (0,1), (1,0) and (1,1) and four corresponding second highest bids, 0, 0, 0, 1 respectively. Therefore, the seller's expected revenue is

$\frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 1 = \frac{1}{4}$

(b)

When there are three bidders, there are eight possible pairs of values (0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0) and (1,1,1) and eight corresponding second highest bids, 0, 0, 0, 1, 0, 1, 1, 1 respectively. Therefore, the seller's expected revenue is

$\frac{1}{8} \cdot 0 + \frac{1}{8} \cdot 0 + \frac{1}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 1 = \frac{1}{8} \cdot 4 = \frac{1}{2}$

(c)

The conjecture is that as the number of bidders increases, the seller's expected revenue increases, in fact, converging to 1. The reason is the following. Notice from part (a) and (b) that if we have at least 2 bidders that bid 1, then the second highest bid is 1, which adds to the seller's revenue.

Intuitively, as the number of bidders increases, the probability that at least 2 bidders bid 1 increases, which increases the seller's revenue.

Let n be the number of bidders.

$$\begin{aligned} \text{P(at least 2 bidders bid 1)} &= 1-\text{P(At most 1 bidder bid 1)} \\ &= 1 - \frac{\binom{n}{0}-\binom{n}{1}}{2^n} \\ &= 1 - \frac{1+n}{2^n} \end{aligned}$$

Therefore, as $n \to \infty$, P(at least 2 bidders bid 1) $\to 1$, which results in the seller's expected revenue approaching 1.

3 Collusion in Auctions

(a)

Since there are only two bidders, one of the two people will win and receive payoff $v_{winner} - bid_{loser}$, while the other loses and receives a payoff of 0.

To maximize the total payoff, we want to maximize v_{winner} and minimize bid_{loser}. Therefore, the optimal strategy is (a) for the person with the lower value to bid 0, thereby providing the lowest possible second price bid_{loser} and (b) for the person with the higher value to bid their own value (or alternatively any amount greater than bid_{loser}), thereby maximizing v_{winner}.

(b)

If there is a third bidder who is not part of the collusion, the existence of this bidder should **not change** the optimal bids for the two bidders. Specifically, the bidder with the higher value should now bid his value, and the bidder with the lower value should bid 0.

Suppose there are three bidders, b_1 with value v_1, b_2 with value v_2 and b_3 with value v_3. Furthermore, let's suppose that b_1 and b_2 are colluding and that $v_2 > v_1$. There are three possible scenarios:

$v_1 < v_2 < v_3$ — b_1 and b_2 do not know v_3 and they do not have the incentive overbid because otherwise they will receive negative payoff. Since the two colluding bidders are guaranteed to lose and receive a maximum payoff of 0, $(0,v_2)$ is the optimal bidding strategy for the colluding bidders in this case.

$v_1 < v_3 < v_2$ — b_3 does not have the incentive to overbid (because of the risk of negative payoff) or underbid (because of the risk of losing), so b_3 will bid his value v_3. b_2 should bid v_2 because regardless of his bid, the maximum payoff is $v_2 - v_3$ anyways. b_1 should bid 0 because he will lose anyways. Therefore, $(0,v_2)$ is the optimal bidding strategy for the colluding bidders in this case.

$v_3 < v_1 < v_2$ — b_3 does not have the incentives to overbid because of the risk of negative payoff, so he will bid his value v_3. b_1 should bid 0 because by doing so, he essentially maximizes the payoff by allowing v_3 to become the second highest bidder and therefore a lower bid_{loser}. Therefore, $(0,v_2)$ is the optimal bidding strategy for the colluding bidders in this case.